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Two Iteration Processes for Computing Positive Definite Solutions of the Equation $X - A^*X^{-n}A = Q$

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Abstract—In the present paper, we suggest two iteration methods for obtaining positive definite solutions of nonlinear matrix equation $X - A^*X^{-n}A = Q$, for the integer $n \geq 1$. We obtain sufficient conditions for existence of the solutions for the matrix equation. Finally, some numerical examples to illustrate the effectiveness of the algorithms and some remarks. © 2001 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

We consider the matrix equation

$$X - A^*X^{-n}A = Q, \tag{1.1}$$

where Q is square positive definite matrix and A is an invertible matrix. Obviously, this equation can be reduced to

$$X - A^*X^{-n}A = I, \tag{1.2}$$

where I is the identity matrix, see [1].

The nonlinear matrix equation (1.2), arises in many applications, for the case $n = 1$. These areas include control theory, ladder networks, dynamic programming, stochastic filtering, and statistic; see references given in [1]. This matrix equation has been studied in recent papers; see [1–4], for various values of n. The matrix equation $X + A^*X^{-1}A = I$ and its general forms are discussed by several authors [5–9]. Ferrante and Levy [1] considered the equation $X = Q + NX^{-1}N^*$, and presented an algorithm which converges to the positive definite solution for a wide range of conditions. Ivanov *et al.* [2] discussed the properties of the positive definite solution of the equation $X - A^*X^{-2}A = I$. El-Sayed and Ramadan [4] studied the existence

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of a positive definite solution of the matrix equation $X + A^* \sqrt[2^m]{X^{-1}} A = I$. Finally, Guo and Lancaster [3] presented many iterative methods of matrix equation (1.1) when $n = 1$ and the matrix equation $X + A^* X^{-1} A = Q$. They discussed the Newton's method and the inversion free variants of the basic fixed-point iteration.

In the present paper, we suggest two iterative methods to obtain a solution of the nonlinear matrix equation (1.2) which has the general form than [1–4]. We obtain the conditions ensure the iterative processes convergent to the solution of equation (1.2).

This paper is organized as follows. In Section 2, we present a first iterative process for obtaining the solution of our problem. Also, it contains theorems for obtaining the sufficient conditions for the existence of a solution of matrix equation (1.2). Section 3 contains the second iterative process for obtaining the solution of the problem and contain theorems for the sufficient conditions for the existence of a positive definite solution of equation (1.2). Numerical examples are given and discussed in Section 4 to illustrate the effectiveness of the methods. Finally, Section 5 contains the conclusions on the results include in the paper.

The notation $X > 0$ means that X is a positive definite Hermitian matrix $A > B$ is used to indicate that $A - B > 0$. Finally, throughout the paper, $\|.\|$ will be the spectral norm for square matrices unless otherwise noted.

2. THE FIRST ITERATION PROCESS

The first iterative method which is suitable for obtaining a positive definite solution of equation (1.2) is discussed in this section. We prove the some properties which will be used throughout this section.

LEMMA 2.1. *(See [7].) If $P \geq Q > 0$, then $P^{-1} \leq Q^{-1}$.*

LEMMA 2.2. *(See [7].) If P and Q are positive definite matrices for which $P - Q > 0$ and $PQ = QP$ are satisfied, then $P^n - Q^n > 0$.*

Consider the sequence of the following matrices:

$$X_0 = I, \quad X_{s+1} = I + A^* X_s^{-n} A, \qquad s = 0, 1, 2, \dots. \tag{2.1}$$

From now to the end of this section, we will take A is normal matrix ($AA^* = A^*A$) for ensure the commute of the matrices X_s generated from sequence (2.1). The following two lemmas are doing this.

LEMMA 2.3. *If the matrix A is normal, then*

$$AX_s = X_s A, \qquad s = 0, 1, 2, \dots, \tag{2.2}$$

where X_s are the matrices generated from sequence (2.1).

PROOF. Since $X_0 = I$, then $AX_0 = X_0 A$. Using the condition $AA^* = A^*A$, we obtain $A + AA^*A = A + A^*AA$ and $AX_1 = X_1 A$.

Let $AX_k = X_k A$ for each $k \leq s$. We want prove that $AX_{k+1} = X_{k+1} A$ for $k \leq s$, thus

$$A + A^* X_k^{-n} AA = A + A^* A X_k^{-n} A \qquad \text{and} \qquad X_{k+1} A = AX_{k+1},$$

which completes the induction.

LEMMA 2.4. *If the matrix A is normal, then*

$$X_{s+1} X_s = X_s X_{s+1}, \qquad s = 0, 1, 2, \dots,$$

where X_s are the matrices from sequence (2.1).

PROOF. Since $X_0 = I$ and $X_1 = I + A^*A$, then $X_0X_1 = X_1X_0$. Further on we compute

$$X_1X_2 = I + A^*A + A^*X_1^{-n}A + A^*AA^*X_1^{-n}A,$$
$$X_2X_1 = I + A^*A + A^*X_1^{-n}A + A^*X_1^{-n}AA^*A.$$

Since A is normal, we get

$$(I + A^*A)\,AA^* = AA^*\,(I + A^*A), \qquad \text{then } X_1AA^* = AA^*X_1.$$

Multiplying both sides of the last equality by the matrix X_1^{n-1} gives

$$X_1^nAA^* = X_1^{n-1}AA^*X_1 = AA^*X_1^n \qquad \text{and} \qquad A^*A\left(I + A^*X_1^{-n}A\right) = \left(I + A^*X_1^{-n}A\right)A^*A.$$

Thus, we have $X_1X_2 = X_2X_1$. This means that the commute is hold for $s = 0, 1$. Now, we assume that for each k it is satisfied

$$X_kX_{k-1} = X_{k-1}X_k. \tag{2.3}$$

Now, we will prove that $X_{k+1}X_k = X_kX_{k+1}$. Since $AX_k = X_kA$, we have

$$AA^*X_k = X_kAA^* \qquad \text{and} \qquad X_k^{-n}AA^* = AA^*X_k^{-n}. \tag{2.4}$$

We shall use the following equality:

$$X_k^nAA^*X_{k-1}^n = X_k^nAA^*X_{k-1}^n.$$

According to (2.2) and (2.3), we have that

$$AA^*X_k^{-n}X_{k-1}^{-n} = X_{k-1}^{-n}X_k^{-n}AA^*.$$

Applying equality (2.4) yields

$$A^*X_k^{-n}AA^*X_{k-1}^{-n}A = A^*X_{k-1}^{-n}AA^*X_k^{-n}A.$$

We get

$$\left(I + A^*X_k^{-n}A\right)\left(I + A^*X_{k-1}^{-n}A\right) = \left(I + A^*X_{k-1}^{-n}A\right)\left(I + A^*X_k^{-n}A\right),$$
$$X_{k+1}X_k = X_kX_{k+1}.$$

In the following, we will investigate the properties of iteration process (2.1).

THEOREM 2.1. *If A is normal matrix and the inequality*

$$\|A\|^2\left(1 + \|A\|^2\right)^{n-1} < \frac{1}{n} \tag{2.5}$$

is satisfied, then equation (1.2) has a positive definite solution, which satisfy

$$X_{2s} < X < X_{2s+1}, \qquad s = 1, 2, \dots$$

and

$$\max\left(\|X - X_{2s}\|, \|X_{2s+1} - X\|\right) < q^{2s}\|AA^*\|,$$

where

$$q = n\|A\|^2\left(1 + \|A\|^2\right)^{n-1} < 1 \qquad \text{and} \qquad s = \left[\frac{(\ln\varepsilon - \ln\|AA^*\|)}{(2\ln q)}\right] + 1.$$

PROOF. We consider the sequence of matrices (2.1). For X_1, we have $X_1 = I + A^* X_0^{-n} A = I + A^* A > I = X_0$. Since $X_0 < X_1$, then $X_0^{-1} > X_1^{-1}$ and

$$A^* X_0^{-n} A > A^* X_1^{-n} A.$$

Hence,

$$I = X_0 < X_2 = I + A^* X_1^{-n} A < I + A^* X_0^{-n} A = X_1,$$

i.e., $X_0 < X_2 < X_1$. To prove $X_s < X_1$ if $X_{s-1} > X_0$ for all s, hence $X_s = I + A^* X_{s-1}^{-n} A < I + A^* A = X_1$.

We find the relation between X_2, X_3, X_4. Since $X_1 > X_2$, then $X_2 = I + A^* X_1^{-n} A < I + A^* X_2^{-n} A = X_3$ and $X_4 = I + A^* X_3^{-n} A < I + A^* X_2^{-n} A = X_3$. Also since $X_3 < X_1$, then $X_4 = I + A^* X_3^{-n} A > I + A^* X_1^{-n} A = X_2$ and $X_5 = I + A^* X_4^{-n} A < I + A^* X_2^{-n} A = X_3$. Thus, we get $X_0 = I < X_2 < X_4 < X_5 < X_3 < X_1 = I + A^* A$.

We will prove that $X_0 < X_{s+1} < X_s$. If we have $X_0 < X_{s-1} < X_s$, thus

$$X_0 = I < I + A^* X_{s-1}^{-n} A > I + A^* X_s^{-n} A \qquad \text{and} \qquad X_0 < X_s > X_{s+1}.$$

Also, we can prove that $X_0 < X_s < X_{s+1}$. If we have $X_0 < X_s < X_{s-1}$, then

$$X_0 = I < I + A^* X_{s-1}^{-n} A < I + A^* X_s^{-n} A \qquad \text{and} \qquad X_0 < X_s < X_{s+1}.$$

Therefore, we have

$$X_0 = I < X_{2r} < X_{2r+2} < X_{2s+3} < X_{2s+1} < X_1 = I + A^* A$$

for every positive integers r, s.

Consequently, the subsequences $\{X_{2r}\}, \{X_{2s+1}\}$ are monotonic and bounded. Therefore, they are convergent to positive definite matrices. To prove these sequences have a common limit, we have

$$\begin{aligned}\|X_{2s+1} - X_{2s}\| &= \left\|A^* X_{2s}^{-n} A - A^* X_{2s-1}^{-n} A\right\| = \left\|A^* X_{2s-1}^{-n} \left(X_{2s-1}^{n} - X_{2s}^{n}\right) X_{2s}^{-n} A\right\| \\ &\le \|A\|^2 \left\|X_{2s-1}^{-n}\right\| \left\|X_{2s}^{-n}\right\| \left\|X_{2s-1}^{n} - X_{2s}^{n}\right\|.\end{aligned}$$

From the identity

$$X_{2s-1}^{n} - X_{2s}^{n} = (X_{2s-1} - X_{2s}) \sum_{j=1}^{n} X_{2s-1}^{n-j} X_{2s}^{j-1},$$

$$\left\|X_{2s-1}^{2} - X_{2s}^{2}\right\| \le \|X_{2s-1} - X_{2s}\| \left(\sum_{j=1}^{n} \left\|X_{2s-1}^{n-j}\right\| \left\|X_{2s}^{j-1}\right\|\right).$$

$$\|X_{2s+1} - X_{2s}\| \le \|A\|^2 \|X_{2s-1} - X_{2s}\| \left(\sum_{j=1}^{n} \|X_{2s-1}\|^{n-j} \|X_{2s}\|^{j-1}\right).$$

Since $I < X_s < I + A^* A$, then we have

$$\left\|X_s^{-n}\right\| < 1 \qquad \text{and} \qquad \|X_s\| < 1 + \|A\|^2.$$

Consequently,

$$\|X_{2s+1} - X_{2s}\| \le n\|A\|^2 \left(1 + \|A\|^2\right)^{n-1} \|X_{2s-1} - X_{2s}\|.$$

Since (2.5), it follows that

$$q = n\|A\|^2 \left(1 + \|A\|^2\right)^{n-1} < 1,$$

and we get

$$\|X_{2s+1} - X_{2s}\| \le q\|X_{2s-1} - X_{2s}\| \le \cdots \le q^{2s}\|X_1 - X_0\|.$$

Therefore, we have for the limit X of sequence (2.1)

$$X_{2s} < X < X_{2s+1}, \qquad s = 1, 2, \dots,$$

and consequently,

$$\max\left(\|X - X_{2s}\|, \|X_{2s+1} - X\|\right) < \|X_{2s+1} - X_{2s}\| < q^{2s}\|X_1 - X_0\| = q^{2s}\|AA^*\| = \varepsilon.$$

From the above inequality, we receive to

$$s = \left[\frac{\ln\varepsilon - \ln\|AA^*\|}{2\ln q}\right] + 1,$$

where $[p]$ means that, the greatest integer not greater than p.

COROLLARY 2.1. *If X is a positive definite solution of equation (1.2), then*

$$I < X < I + A^*A.$$

COROLLARY 2.2. *If λ is an eigenvalue of a positive definite solution X of equation (1.2), then*

$$1 < \lambda < 1 + \rho(A^*A),$$

where $\rho(A^*A)$ is the spectral radius of the matrix A^*A.

THEOREM 2.2. *Let X be a solution of (1.2), and denote by μ_+ and μ_- the largest and smallest eigenvalue of X, respectively. If λ is an eigenvalue of A, then*

$$\sqrt{\mu_-^n(\mu_- - 1)} \le |\lambda| \le \sqrt{\mu_+^n(\mu_+ - 1)}.$$

PROOF. Let v be an eigenvector corresponding to an eigenvalue λ of the matrix A and $|v| = 1$. Then

$$\langle Xv, v\rangle - \left\langle A^*X^{-n}Av, v\right\rangle = \langle v, v\rangle,$$
$$|\lambda|^2\left\langle X^{-n}v, v\right\rangle = \langle (X - I)v, v\rangle.$$

Now $X - I$ is positive definite and $(\mu_- - 1)I \le X - I \le (\mu_+ - 1)I$, and the largest eigenvalue of X^{-n} is μ_-^{-n}, so

$$|\lambda|^2(\mu_-)^{-n} \ge \mu_- - 1.$$

Likewise, as the smallest eigenvalue of X^{-n} is μ_+^{-n}, we have that

$$|\lambda|^2(\mu_+)^{-n} \le \mu_+ - 1.$$

3. THE SECOND ITERATION PROCESS

In this section, we will establish the second iterative method which also is suitable for obtaining to a positive definite solution of equation (1.2).

$$X_0 = \alpha I, \quad X_{s+1} = \sqrt[n]{A(X_s - I)^{-1}A^*}, \qquad s = 0, 1, 2, \dots. \tag{3.1}$$

LEMMA 3.5. *If the $A > B > 0$, then $\sqrt[2^m]{A} > \sqrt[2^m]{B} > 0$.*

PROOF. See [4].

Now we discuss the convergent for the sequence generated from process (3.1).

THEOREM 3.3. *If there is a real α so that $\alpha > 1$ and the following:*

(i) $AA^* < \alpha^n(\alpha - 1)I$,
(ii) $\sqrt[n]{(AA^*)/(\alpha - 1)} - (1/\alpha^n)A^*A > I$,
(iii) $\|A\|^2 < 2^m \mu^{5/2} / \alpha^{2^{2m+1}} \sqrt{\alpha - 1}$

are satisfied, where $n = 2^m$ and μ is the smallest eigenvalue of the matrix AA^*, then equation (1.2) has a positive definite solution.

PROOF. We consider sequence (3.1). For X_1, we have

$$X_1 = \sqrt[n]{A(X_0 - I)^{-1}A^*} = \sqrt[n]{\frac{AA^*}{\alpha - 1}} < \alpha I = X_0.$$

Hence, $X_1 < X_0$. Applying Condition (ii), we obtain

$$X_1 = \sqrt[n]{\frac{AA^*}{\alpha - 1}} > I + \frac{1}{\alpha^n}A^*A > I, \qquad \text{then } I < X_1 < X_0.$$

Further on, we have

$$X_2 = \sqrt[n]{A(X_1 - I)^{-1}A^*} > \sqrt[n]{A(X_0 - I)^{-1}A^*} = X_1.$$

Hence, $X_1 < X_2$. Using Condition (ii), we have

$$X_1 - I > \frac{1}{\alpha^n}A^*A \qquad \text{and} \qquad X_2 = \sqrt[n]{A(X_1 - I)^{-1}A^*} < \alpha I = X_0.$$

Then, $I < X_1 < X_2 < X_0$.

Hence, as Theorem 2.1, the general case will be

$$I < X_1 < X_{2s+1} < X_{2s+3} < X_{2k+2} < X_{2k} < X_0 = \alpha I.$$

for every positive integers s, k.

Therefore, the subsequences $\{X_{2k}\}, \{X_{2s+1}\}$ are convergent ones to positive definite matrices. These sequences have a common limit. We have

$$\|X_{2k} - X_{2k+1}\| = \left\| \sqrt[n]{A(X_{2k-1} - I)^{-1}A^*} - \sqrt[n]{A(X_{2k} - I)^{-1}A^*} \right\|. \tag{3.2}$$

We denote

$$P = A(X_{2k-1} - I)^{-1}A^*, \qquad Q = A(X_{2k} - I)^{-1}A^*.$$

We use the following equality:

$$\sqrt[n]{P}\left(\sqrt[n]{P} - \sqrt[n]{Q}\right) + \left(\sqrt[n]{P} - \sqrt[n]{Q}\right)\sqrt[n]{Q} = \left(\sqrt[n/2]{P} - \sqrt[n/2]{Q}\right).$$

Since $X_{2k-1} < X_{2k}$ for each $k = 0, 1, \ldots$, then the matrix $Y = \sqrt[n]{P} - \sqrt[n]{Q}$ is a positive definite solution of the matrix equation

$$\sqrt[n]{P}\, Y + Y \sqrt[n]{Q} = \left(\sqrt[n/2]{P} - \sqrt[n/2]{Q} \right).$$

According to [7, Theorem 3], we have

$$Y = \int_0^{\infty} e^{-\sqrt[n]{P}t} \left(\sqrt[n/2]{P} - \sqrt[n/2]{Q} \right) e^{-\sqrt[n]{Q}t}\, dt. \tag{3.3}$$

Since $\sqrt[n]{P}$, $\sqrt[n]{Q}$ are positive definite matrices then integral (3.3) exists and

$$e^{-\sqrt[n]{P}t} \left(\sqrt[n/2]{P} - \sqrt[n/2]{Q} \right) e^{-\sqrt[n]{Q}t} \to 0, \qquad t \to \infty.$$

Then

$$\|X_{2k} - X_{2k+1}\| \le \int_0^{\infty} \left\| \sqrt[n/2]{P} - \sqrt[n/2]{Q} \right\| \left\| e^{-\sqrt[n]{P}t} \right\| \left\| e^{-\sqrt[n]{Q}t} \right\| dt.$$

However, $X_s < \alpha I$, hence

$$\sqrt[n]{A(X_s - I)^{-1}A^*} > \sqrt[n]{\frac{\mu I}{\alpha - 1}}.$$

Then

$$\begin{aligned} \|X_{2k} - X_{2k+1}\| &\le \left\| \sqrt[n/2]{P} - \sqrt[n/2]{Q} \right\| \int_0^{\infty} e^{-2\sqrt[n]{\mu/(\alpha-1)}t}\, dt, \\ &= \frac{1}{2} \left\| \sqrt[n/2]{P} - \sqrt[n/2]{Q} \right\| \sqrt[n]{\frac{\alpha - 1}{\mu}} \le \frac{1}{4} \left\| \sqrt[n/4]{P} - \sqrt[n/4]{Q} \right\| \sqrt[n/2]{\frac{\alpha - 1}{\mu}}. \end{aligned}$$

After m times as above, we get

$$\|X_{2k} - X_{2k+1}\| \le \frac{1}{2^m} \left\| \sqrt[n/2^m]{P} - \sqrt[n/2^m]{Q} \right\| \sqrt[n/2^{m-1}]{\frac{\alpha - 1}{\mu}}.$$

Since $n = 2^m$ is special case, then we have

$$\begin{aligned} \|X_{2k} - X_{2k+1}\| &\le \frac{1}{2^m} \|P - Q\| \sqrt[2^m/2^{m-1}]{\frac{\alpha - 1}{\mu}}, \\ &= \frac{1}{2^m} \sqrt{\frac{\alpha - 1}{\mu}} \left\| A(X_{2k-1} - I)^{-1}A^* - A(X_{2k} - I)^{-1}A^* \right\| \\ &= \frac{1}{2^m} \sqrt{\frac{\alpha - 1}{\mu}} \left\| A(X_{2k} - I)^{-1} (X_{2k} - X_{2k-1}) (X_{2k-1} - I)^{-1}A^* \right\|. \end{aligned}$$

Since we have $X_1 < X_s$ for the matrices of sequence (3.1), hence

$$\frac{AA^*}{\alpha^n} < X_1 - I < X_s - I,$$

$$(X_s - I)^{-1} < \alpha^n (AA^*)^{-1}, \qquad \text{and} \qquad \left\| (X_s - I)^{-1} \right\| < \frac{\alpha^n}{\mu}.$$

Thus, we get

$$\|X_{2k} - X_{2k+1}\| \le \frac{1}{2^m} \sqrt{\frac{\alpha - 1}{\mu}} \|A\|^2 \frac{\alpha^{2^{m+1}}}{\mu^2} \|X_{2k} - X_{2k-1}\|.$$

Condition (iii) implies

$$q = \frac{\alpha^{2^{m+1}}}{2^m \mu^2} \|A\|^2 \sqrt{\frac{\alpha - 1}{\mu}} < 1,$$

and then,

$$\|X_{2s+1} - X_{2s}\| \le q\|X_{2s-1} - X_{2s}\| \le \cdots \le q^{2s}\|X_1 - X_0\|.$$

We have for the limit X of sequence (3.1)

$$X_{2s} < X < X_{2s+1}, \qquad s = 1, 2, \dots,$$

and we get

$$\max\left(\|X - X_{2s}\|, \|X_{2s+1} - X\|\right) < \|X_{2s+1} - X_{2s}\| < q^{2s}\|X_1 - X_0\| = q^{2s}\|AA^*\|.$$

Therefore, the subsequences $\{X_{2k}\}, \{X_{2s+1}\}$ are convergent ones and have a common positive definite limit which is a solution of the matrix equation (1.2).

We prove the above theorem, but with another Condition (iii).

THEOREM 3.4. *If there is a real α such that $\alpha > 1$ and the two Conditions (i) and (ii) in Theorem (3.2) are satisfied and also*

(iii)$'$ $(\alpha^{2^{2m-2}}(\alpha - 1)^{(2^m-1)/2})/2^m < 1$,

satisfies, then equation (1.2) has a positive definite solution.

PROOF. The proof of this theorem have the sane manner as the above theorem, but from equation (3.2), we get

$$\begin{aligned}\|X_{2k} - X_{2k+1}\| &= \left\| \sqrt[n]{A(X_{2k-1} - I)^{-1}A^*} - \sqrt[n]{A(X_{2k} - I)^{-1}A^*} \right\|, \\ &< \|AA^*\|^{1/n} \left\| \sqrt[n]{(X_{2k-1} - I)^{-1}} - \sqrt[n]{(X_{2k} - I)^{-1}} \right\|,\end{aligned}$$

Here we take

$$P = (X_{2k-1} - I)^{-1}, \qquad Q = (X_{2k} - I)^{-1}.$$

Similar to Theorem 3.2, we get

$$\begin{aligned}\|X_{2k} - X_{2k+1}\| &\le \|AA^*\|^{1/n} \left\| \int_0^\infty e^{-\sqrt[n]{P}t} \left(\sqrt[n/2]{P} - \sqrt[n/2]{Q} \right) e^{-\sqrt[n]{Q}t}\, dt \right\| \\ &\le \|AA^*\|^{1/n} \int_0^\infty \left\| \sqrt[n/2]{P} - \sqrt[n/2]{Q} \right\| \left\| e^{-\sqrt[n]{P}t} \right\| \left\| e^{-\sqrt[n]{Q}t} \right\| dt.\end{aligned}$$

However, $X_s < \alpha I$, thus

$$\sqrt[n]{(X_s - I)^{-1}} > \sqrt[n]{\frac{I}{\alpha - 1}}.$$

Therefore, we get

$$\begin{aligned}\|X_{2k} - X_{2k+1}\| &\le \|A\|^{2/n} \left\| \sqrt[n/2]{P} - \sqrt[n/2]{Q} \right\| \int_0^\infty e^{-2\sqrt[n]{1/(\alpha-1)}t}\, dt, \\ &= \frac{1}{2}\|A\|^{2/n} \left\| \sqrt[n/2]{P} - \sqrt[n/2]{Q} \right\| \frac{1}{\sqrt[n]{\alpha - 1}}.\end{aligned}$$

Since $n = 2^m$, then after m times, we get

$$\|X_{2k} - X_{2k+1}\| \le \frac{\|A\|^{2^{(1-m)}}}{2^m \sqrt{\alpha - 1}(\alpha - 1)^2} \|X_{2k} - X_{2k-1}\|.$$

From the second condition, we get $\|A\|^2 < \alpha^{2^m}(\alpha - 1)$. Then we have

$$\|X_{2k} - X_{2k+1}\| \le \frac{\alpha^{2^{2m-2}}(\alpha - 1)^{(2^m-1)/2}}{2^m} \|X_{2k} - X_{2k-1}\|.$$

Condition (iii) implies

$$q = \frac{\alpha^{2^{2m-2}}(\alpha - 1)^{(2^m-1)/2}}{2^m} < 1.$$

Consequently, the subsequences $\{X_{2k}\}, \{X_{2s+1}\}$ are convergent and have a common positive definite limit which is a solution of the matrix equation (1.2).

4. NUMERICAL RESULTS

In this section, we report some numerical results, that describe the performance of the algorithms. The following tables indicate the convergence pattern of the iterative sequences of approximate solution. The solution is computed for different matrices A and different values of n. We denote $\varepsilon(Z) = \|Z + A^\top Z^{-2}A - I\|_\infty$, X the solution which is obtained by the iterative method (2.1) and denote m_X be the smallest number s, for which

$$\|X_s - X\| \le \left(n\,\|A\|^2\left(1+\|A\|^2\right)^{n-1}\right)^s \|AA^*\| \le 10^{-5}.$$

Also denote Y be the solution which is obtained by the iterative method (3.1) and m_Y be the smallest number k, for which

$$\|Y_k - Y\| \le \left(\frac{\alpha^{2^{m+1}}}{2^m\mu^2}\|A\|^2\sqrt{\frac{\alpha-1}{\mu}}\right)^k \le 10^{-5}.$$

EXAMPLE 1. Let $n = 2$ and A be from the type

$$A = (a_{ij}) = \begin{cases} a_{ij} = \dfrac{i}{n^4+100}, & i = j, \\[2ex] a_{ij} = \dfrac{j}{10n^2+100}, & i \ne j. \end{cases}$$

Table 1.

n	m_X	m_Y	$\varepsilon(X_s)$	$\varepsilon(Y_p)$
2	6	4	1.0353E − 8	1.9048E − 5
6	13	4	2.4288E − 10	1.0428E − 5
10	17	5	4.8012E − 11	9.9482E − 6
14	19	5	2.4328E − 12	9.7008E − 6
18	20	5	1.3919E − 12	9.3478E − 6
22	20	5	1.8189E − 12	1.4012E − 7
26	21	5	1.8189E − 12	1.4009E − 7

EXAMPLE 2. Let $n = 2^3$ and A be from the type

$$A = (a_{ij}) = \begin{cases} a_{ij} = \dfrac{i^{1/4}}{i^4+10}, & i = j, \\[2ex] a_{ij} = 0.005, & i \ne j. \end{cases}$$

Table 2.

n	m_X	m_Y	$\varepsilon(X_s)$	$\varepsilon(Y_p)$
2	8	10	8.0857E − 6	2.9847E − 8
6	8	10	4.0138E − 5	3.6176E − 9
10	6	8	9.7035E − 5	5.6351E − 10
14	6	8	3.4701E − 4	4.3304E − 10
18	7	8	4.9689E − 4	4.0164E − 10
22	7	8	7.204E − 4	6.9371E − 12
26	7	8	8.9405E − 4	1.0416E − 12

5. CONCLUSION

We considered a nonlinear matrix equation which in general form than [1–4]. Also, we achieved general conditions for the existence of a positive definite solution. Moreover, we discussed two iterative algorithms from which a solution can always be calculated numerically, whenever the equation is solvable. We compared between the two iterations with two examples. These numerical results show the efficiency of the two iterative methods which are described. We note that the results obtained so far on the iterative procedures for finding positive definite solutions, are more general than those obtained in [1–4], in the sense that we deal with a large class of matrix equations. The equation has in general more than one solution. Questions arise, whether all solutions can be ordered and whether there exist a smallest and a largest solution. The answers of these questions are still a topic of future research.

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